**Average Value**

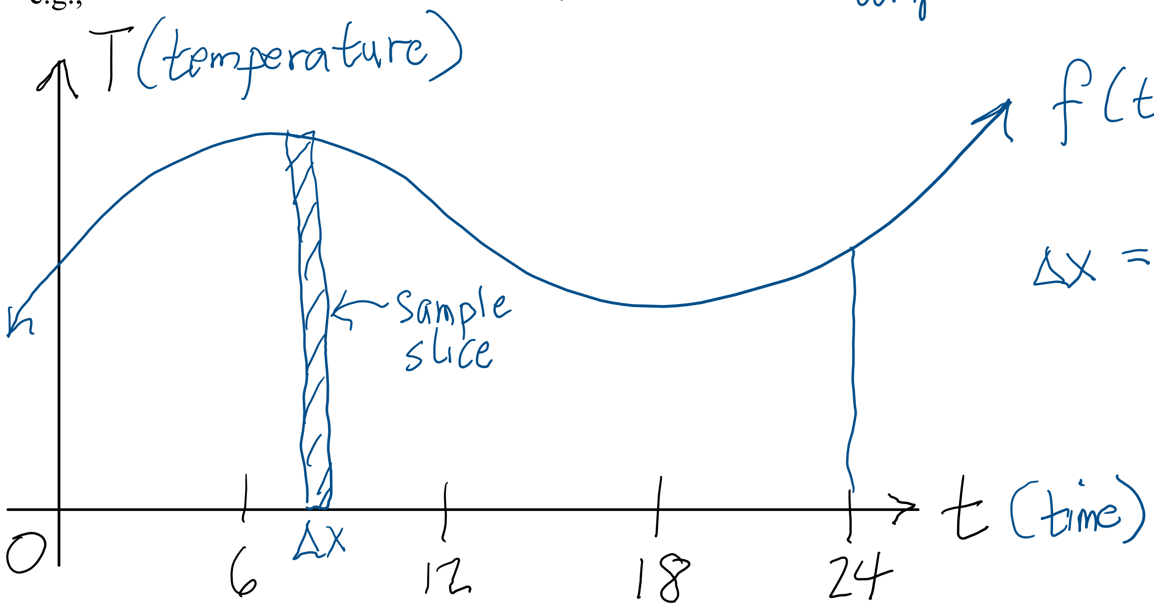
With a finite list of numbers, to find its average/mean:

$$y_{AVG} = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

where $n$ represents the number of values being averaged

With infinitely many numbers, need a new plan.

e.g., ex. continuous temperature monitoring




$\Delta x = \frac{b-a}{n}$

↓ solve for n

$n = \frac{b-a}{\Delta x}$

$$\text{avg value} = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{\cancel{\frac{b-a}{\Delta x}}} \cdot \frac{\Delta x}{b-a} \quad \text{assuming } n \text{ slices}$$

KCF

$$= \frac{1}{b-a} \cdot \underbrace{\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right]}_{\text{write in sigma notation}} \cdot \Delta x$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \cdot \Delta x \quad \text{finite \# of slices}$$

for infinitely many slices: $= \frac{1}{b-a} \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$

$$f_{AVG} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

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ex. find average value of $f(x) = 1+x^2$ on $[-1, 2]$ ($a = -1$, $b = 2$)

$$f_{AVG} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx$$
$$= \frac{1}{3}\left(x+\frac{x^3}{3}\right)\Big|_{-1}^{2}$$
$$= \frac{1}{3}\left((2-(-1)) + \frac{8}{3} - \frac{(-1)^3}{3}\right)$$
$$= \frac{1}{3}\left(3 + \frac{8}{3} + \frac{1}{3}\right)$$
$$= \frac{1}{3}\left(3 + \cancel{\frac{9}{3}}^{\,3}\right) = \frac{1}{3}(6) = \boxed{2}$$

u-sub !!

Do: find average value of $f(x) = \cos^4 x \sin x$ on $[0, \pi]$

$$f_{AVG} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{\pi-0}\int_0^{\pi}\cos^4 x \sin x\,dx$$

$u = \cos x$

$-du = \sin x\,dx$

change bounds

$u_a = \cos 0 = 1$

$u_b = \cos\pi = -1$

$$= -\frac{1}{\pi}\int_1^{-1} u^4\,du \qquad = -\frac{1}{\pi}\int_{x=0}^{x=\pi} u^4\,du$$
$$= -\frac{1}{5\pi}\,u^5\Big|_1^{-1} \qquad -\frac{1}{5\pi}\,u^5\Big|_{x=0}^{x=\pi}$$
$$= -\frac{1}{5\pi}\left((-1)^5 - 1^5\right) \qquad = -\frac{1}{5\pi}(\cos x)^5\Big|_0^{\pi}$$
$$= -\frac{1}{5\pi}(-1-1) = -\frac{1}{5\pi}(-2) = \boxed{\frac{2}{5\pi}}$$

ex. find average value of $f(x) = xe^x$ on $[1,2]$

$$f_{AVG} = \frac{1}{2-1}\int_1^2 xe^x\,dx \quad \text{USE IBP} \quad \begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}$$

$uv - \int v\,du$

$$= (1)\left(xe^x\Big|_1^2 - \int_1^2 e^x\,dx\right)$$

$$= 2e^2 - e - e^x\Big|_1^2$$

$$= 2e^2 - e - (e^2 - e)$$

$$= 2e^2 - \cancel{e} - e^2 + \cancel{e}$$

$$= \boxed{e^2}$$

$$f_{AVG} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Do: find average value of $f(x) = \sqrt[3]{x}$ on $[1,8]$

$$f_{AVG} = \frac{1}{8-1}\int_1^8 x^{1/3}\,dx$$

$$= \frac{1}{7}\cdot\frac{3}{4}\, x^{4/3}\Big|_1^8$$

$$= \frac{3}{28}\left((8^{1/3})^4 - 1\right)$$

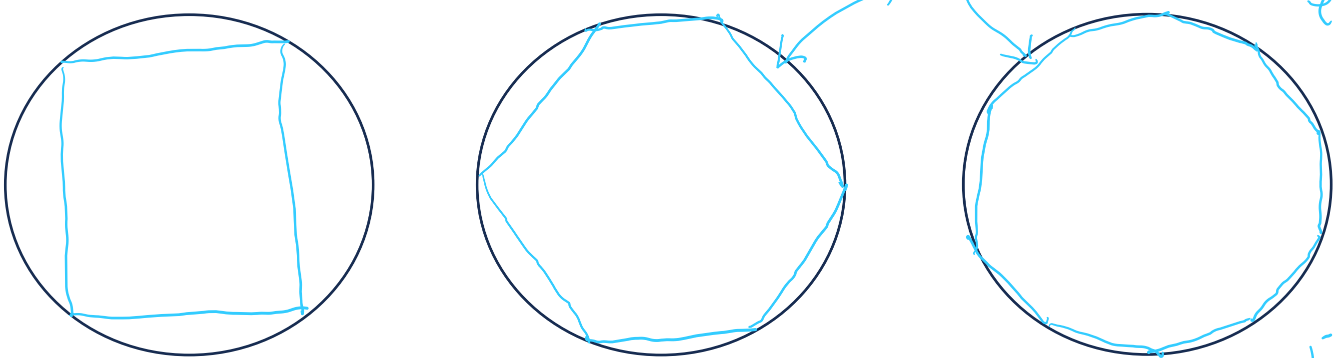
$$= \frac{3}{28}\left(2^4 - 1\right)$$

$$= \frac{3}{28}(16-1) = \frac{3}{28}\cdot 15 = \boxed{\frac{45}{28}}$$

**Arc Length**

to determine the circumference of a circle, first estimate using inscribed polygons

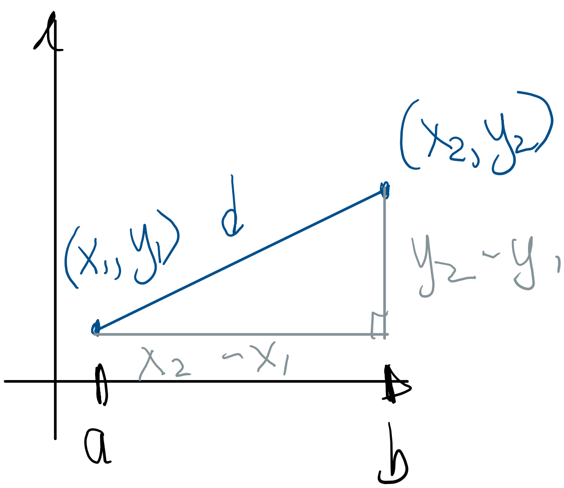
using straight lines for estimate



SQUARE

HEXAGON (better estimate)

even better estimate of circle's circumference




using Pythagorean Theorem

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\sqrt{d^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$L_{\text{"CURVE"}} = d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

for now, these are changing WRT t

⋮

$L_{\text{CURVE}}$ = sum of all (infinitely many) lines

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\Delta x \to \dfrac{dx}{dt}$

$\Delta y \to \dfrac{dy}{dt}$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

GIVEN ONE FUNCTION WRT X ONE FUNCTION WRT y ex. setup the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$

use y for bounds

$\frac{d}{dy}(x = y^2)$ ← WRT y

$\left(\frac{dx}{dy}\right)^2 = (2y)^2$

$= 4y^2$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

$$= \boxed{\int_0^1 \sqrt{4y^2 + 1}\, dy}$$

difficult to integrate by hand

ex: set up length of curve for $y = x^3$ on $[2,5]$

$\frac{dy}{dx} = 3x^2$

$$L = \int_2^5 \sqrt{1 + (3x^2)^2}\, dx$$

$$= \boxed{\int_2^5 \sqrt{1 + 9x^4}\, dx}$$

ex.: find exact length of curve for $x = y^{3/2}$ on $0 \le y \le 1$

$\left(\frac{dx}{dy}\right)^2 = \left(\frac{3}{2} y^{1/2}\right)^2$

$\left(\frac{dx}{dy}\right)^2 = \frac{9}{4} y$

$$L = \int_0^1 \sqrt{\frac{9}{4} y + 1}\, dy$$

$u = \frac{9}{4} y + 1$

$du = \frac{9}{4} dy$

$\frac{4}{9} du = dy$

$u_{y=0} = 1$

$u_{y=1} = \frac{9}{4} + \frac{4}{4} = \frac{13}{4}$

$$= \frac{4}{9} \int_1^{13/4} u^{1/2}\, du$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{13/4}$$

$$= \boxed{\frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)}$$

ex. find exact length of curve for $y = 2x$ $1 \leq x \leq 9$

$\frac{dy}{dx} = 2$

$\left(\frac{dy}{dx}\right)^2 = 4$

$$L = \int_1^9 \sqrt{1+4}\, dx$$
$$= \int_1^9 \sqrt{5}\, dx$$
$$= \sqrt{5}x \Big|_1^9$$
$$= \sqrt{5}(9-1) = \boxed{8\sqrt{5}}$$